Unimodular $\varepsilon$-Hermitian Forms Revisited

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Let $K$ be an algebraic number field with a nontrivial involution, and let $S$ be a Dedekind set of primes of $K$ (cf. [9, Sect. 21]) which is invariant under this involution. Let $A$ be the set of $S$-integers of $K$. We shall study the classification, up to isometry, of unimodular $\varepsilon$-hermitian forms (where $\varepsilon = \pm 1$) on projective $A$-modules of finite rank. This problem has been considered in [1] in the special case where $A$ is the ring of integers of $K$ (i.e., $S$ contains every finite prime of $K$). However, the more general situation naturally arises in knot theoretical problems (see, e.g., Levine [8]).

In the present note we shall see that not only the results of [1] generalize to $S$-integers, but in fact the case where some finite prime of $K$ does not belong to $S$ is much simpler. Every form of rank greater than two behaves like an indefinite form (even if all the signatures are maximal) and a complete set of invariants is given by rank, signatures, terminant (which is a rank one form), and, in the skew-hermitian case, a finite set of pfaffians. The proof of this uses a generalization, due to Kneser, of the strong approximation theorem of G. Shimura, and also some results of Wall.

Let $F$ be the fixed field of the involution, and let $S_0$ be the set of primes $p$ of $F$ such that $p = P \cap F$ for some $P$ in $S$. Assume that $S_0$ contains almost all finite primes of $F$. Let $\Omega_0$ denote the set of all primes, finite and infinite, of $F$. Let us denote $F_p$ the completion of $F$ at $p$, $B_p$ the ring of integers of $F_p$, $K_p = K \otimes_F F_p$ and $A_p = AB_p$.

Let $(V, h)$ be a nonsingular hermitian or skew-hermitian form. We shall say that $S_0$ (or $S$) is an indefinite set of primes for $(V, h)$ if there exists at least one prime $p$ in $\Omega_0 \setminus S_0$ such that $(V, h)_p = (V, h) \otimes_K K_p$ is isotropic (i.e.,

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there exists a nonzero \( x \) in \( V_p \) such that \( h(x, x) = 0 \). A lattice \( L \) in \( (V, h) \) is a finitely generated projective \( A \)-module such that \( L \otimes_A K = V \) and that the restriction of \( h \) to \( L \) takes values in \( A \).

The following is a consequence of a result of Kneser (cf. [5, Satz 2; 6]).

**Theorem 1.** Let \( (V, h) \) be a nonsingular hermitian or skew-hermitian form. Assume that \( \dim(V) > 1 \) and that \( S \) is an indefinite set of primes for \( (V, h) \). Then an \( SU \)-genus of \( A \)-lattices consists of only one \( SU \)-class.

This generalizes Shimura’s theorem [10, 5.19]. One can also use Shimura’s proof, but instead of applying Eichler’s theorem [2, Satz 5] one has to apply a generalization of this theorem (cf. [11, Proposition 5.8]).

Let \( (L, h) \) be a lattice. The determinant of \( (L, h) \) is the rank one form \( \det(L, h) : A^n L \times A^n L \to A \), \( \det(L, h)(x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n) = \det(h(x_i, y_j)) \), where \( n = \text{rank}_A(L) \).

We shall say that \( (L, h) \) is unimodular if the adjoint of \( h \), \( \text{ad}(h) : L \to \text{Hom}_A(L, A) \), given by \( \text{ad}(h)(x)(y) = h(y, x) \), is bijective.

Assume that there exists \( \alpha \in A \) such that \( \alpha + \bar{\alpha} = 1 \) (this hypothesis is satisfied in the knot theoretical applications). This implies that no dyadic prime of \( F \) ramifies in \( K \) (cf. [1, Remark 3.13]).

Let \( (L, h) \) be a unimodular, skew-hermitian lattice of even rank and let \( p \) be a prime of \( F \) which ramifies in \( K \), \( P^2 = pA \). Then the involution on \( A/P \) is trivial (cf. [4, Sect. 5]). The skew-hermitian form \( h \) induces a nonsingular skew-symmetric form \( \bar{h} \) on \( L - L/PL \). Let us denote \( Pf_p(L, h) \) a pfaffian of this form. If \( (M, h) \) is another lattice, and if \( \varphi : L \to M \) is an isometry, then \( Pf_p(M, h) = Pf_p(L, h) \).

The unimodular lattices for which \( S \) is an indefinite set of primes are classified by rank, signatures, determinant, and pfaffians.

**Theorem 2.** Assume that \( S \) is an indefinite set of primes for the unimodular, \( \varepsilon \)-hermitian lattices \( L \) and \( M \). Then \( L \) and \( M \) are isometric if and only if one of the following holds:

(a) \( \varepsilon = +1 \), \( L \), and \( M \) have same rank, signatures, and isometric determinants.

(b) \( \varepsilon = -1 \), \( L \), and \( M \) have same rank, signatures, and there exists an isometry \( f \) between \( \det(L) \) and \( \det(M) \) such that \( \det(f) \ Pf_p(L) \equiv Pf_p(M) \mod P \) for all primes \( p \) of \( F \) such that \( pA \equiv P^2 \).

This is a generalization of [1, Corollary 4.10].
Remark 1. If $S$ does not contain all finite primes of $K$ and if $\dim V > 2$ (where $V = L \otimes_A K$), then the hypothesis of the theorem is always satisfied. Indeed, $(V, h)_p$ is isotropic when $p$ is a finite prime and $\dim V > 2$. Moreover, if $p$ is split (i.e., $pA = P\overline{P}$, where $P \neq \overline{P}$), then $(V, h)_p$ is isotropic also for $\dim(V) = 2$ (notice that $K_p = F_p \times F_p$ if $p$ is split). Therefore the theorem holds in all dimensions provided $\Omega_0 \setminus S_0$ contains a split finite prime.

The following lemma can be deduced from Wall's results (cf. [12]).

**Lemma.** Let $x \in A_p$, $x\bar{x} = 1$. Then there exists a $\psi \in U(V, h)_p$ such that $\det \psi = x$, $\psi(L_p) = L_p$, if and only if either $\varepsilon = 1$, or $\varepsilon = -1$, and $x \equiv 1 \mod P$, where $pA = P^2$.

**Sketch of Proof.** If $\varepsilon = 1$ or $\varepsilon = -1$ and $p$ is unramified, it is easy to obtain the lemma from the classification of unimodular forms over $A_p$ (cf. [10, 4.18; 12, pp. 431-433]). Let us assume that $\varepsilon = -1$ and that $p$ is ramified. Then $(L, h)_p$ is hyperbolic (cf. [12, p. 234; 4, Proposition 8.1.1]). Set $\tilde{L} = L/PL$. Then $\tilde{L}$ supports a nonsingular skew-symmetric form $\tilde{h}$. Let $\psi \in U(V, h)_p$ such that $\psi(L_p) = L_p$. Then $\psi$ induces an automorphism of $(\tilde{L}, \tilde{h})$, the determinant of which must be $+1$, therefore $x = \det(\psi) \equiv 1 \mod P$. Conversely, if $x \equiv 1 \mod P$, then $x = y^2$ with $y \in A_p$ by Hensel's lemma ($p$ is nondyadic) and $y \equiv \pm 1 \mod P$. This, together with $x\bar{x} = 1$ implies $y\bar{y} = 1$. Let $e, f \in L$ be the basis of a hyperbolic plane $H \subset L$. Let us define $\psi(e) = ye, \psi(f) = yf$, and let $\psi$ be the identity on the orthogonal complement of $H$. Then $\psi \in U(V, h)_p$, $\psi(L_p) = L_p$ and $\det(\psi) = x$.

**Proof of Theorem 2.** The conditions of the theorem are clearly necessary. Let us prove that they are also sufficient. By Landherr's theorem (cf. [7]) we can assume that $L$ and $M$ are both lattices in $(V, h)$. By [12, Proposition 6] this implies that $L$ and $M$ are in the same genus. We shall now use a similar argument to Shimura's proof of [10, Proposition 5.27]. Let $f: \det(L) \to \det(M)$ be an isometry and let $a = \det(f)$. Then $aa = 1$. There exists an element $\psi$ of $U(V, h)$ such that $\det \psi = a$.

Let $N = \psi(L)$. For every prime $p$ of $F$ there exists an element $\phi_p$ of $U(V, h)_p$ such that $\phi_p(M_p) = N_p$. On the other hand the existence of $f$ implies that we have an element $F$ of $GL(V)$ such that $F(L) = M$ and $\det(F) = a$. This implies that $\det(\phi_p)$ is a unit for all $p$. We also have $P_f(N) = P_{\phi_p}(M)$ if $p$ is ramified, therefore $\det(\phi_p) = 1 \mod P$, where $P^2 = pA$. By the lemma this implies that there exists an element $\phi$ of $U(V, h)$ such that $\phi_p(M_p) = M_p$ and that $\det(\phi_p) = \det(\phi)^{-1}$. Therefore $N$ and $M$ are in the same $SU$-genus, so they are $SU$-equivalent by Theorem 1.

**Remark 2.** It is easy to check that the other results of [1, Section 4] can also be generalized to the case of $S$-integers. So we have class number formulas (Proposition 4.8, of course one has to replace $C_k$ by the group of
classes of $A$-ideals), decomposition theorems into lattices of rank at most 2 if $\varepsilon = \pm 1$ (Proposition 4.11) and at most 4 if $\varepsilon = -1$ (Proposition 4.12) and cancellation (Proposition 4.13).

Notice that L. Gerstein's decomposition theorem, for non necessarily unimodular lattices, also generalizes to $S$-integers: if $S$ does not contain all finite primes of $K$, then every hermitian $A$-lattice is isometric to the orthogonal sum of lattices of rank at most 4. The proof is as in [3, Theorem 3.14], except that one has to apply Theorem 1 instead of Shimura's theorem.

References